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Thermodynamically consistent description of heat conduction with finite speed of heat propagation

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Abstract

In this work, governing equations for heat conduction with finite speed of heat propagation are derived directly from classical thermodynamics. For a one-dimensional flow of heat, the developed governing equation is linear and of parabolic type. In a three dimensional case, the system of nonlinear equations is formulated.

Analytical solutions of the equations for one-dimensional flow of heat are obtained, and their analysis shows characteristic features of heat propagation with finite speed, being fully consistent with classical thermodynamics.

Keywords: Thermodynamics; Heat conduction; Speed of heat propagation

1. Introduction

In heat conduction, two different analytical approaches have been developed. The first is based on a parabolic type Fourier equation assuming infinite speed of heat propagation. The second assumes finite speed of heat propagation and uses a hyperbolic type governing equation.

Though from molecular point of view, finite speed of heat propagation seems to be a reasonable physical assumption, for years thermodynamically consistent Fourier equation has been successfully applied. In the last fifty years, different versions of the hyperbolic heat conduction equation assuming finite thermal propagation speed have been introduced [1–5]. Unfortunately, their thermodynamic consistency in the frame of classical thermodynamics is not clearly proven despite numerous attempts to obtain such a proof [6–8]. Moreover, some solutions of hyperbolic heat conduction equations analyzed by Taitel [9] and Haji-Sheikh et al. [10] display heat flows from cold to hot bodies, which contradict classical formulation of the Second Law of thermodynamics. Such a situation motivated different authors to publish works revising classical thermodynamics with the aim to justify hyperbolic heat conduction [2,11–13]. These works lead far enough from classical thermodynamics, and describe hyperbolic heat conduction as a consequence of extended new basic thermodynamic equations.

Naturally arises a question, whether classical thermodynamics is principally limited in such a way that finite speed of heat propagation contradicts its basic laws, or a solution of the problem may be found in the frame of classical thermodynamics. In the sequel, heat conduction equations with finite speed of heat propagation fully consistent with classical thermodynamics, are derived, and their properties are studied. The problem is solved purely deductively, by applying the thermodynamic method. According to it, the governing equations for irreversible transport phenomena are derived as corollaries of the First and Second Laws of thermodynamics, and the local thermodynamic equilibrium principle. Such a method has been successfully used for analyses of different transport phenomena such as Fourier type heat conduction [14,15], viscosity, flow of fluids in porous media, diffusion, and energy dissipation associated with direct electric current [16]. In this work, such a thermodynamic method is developed for a case of finite speed of heat propagation in heat conduction.

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Nomenclature

A, B, E constant volume thermodynamic bodies
$A_m = -4/(2m-1)\pi$ Fourier series coefficient
$a = c\tau/l$ dimensionless time for a case of one-dimensional flow of heat in finite region
C_v specific heat capacity
c thermal mediator speed
c_{gr} group velocity
c_{ph} phase velocity
D, D_1, D_2 thermal mediators
D_m coefficient defined by formulae (54) and (56)
K, K_c thermal conductivity
k wave number
l, l_1, l_2 distance, length
m integer
$\mathbf{n} = \nabla T/|\nabla T|$ temperature gradient unit vector
Q heat flow
$\mathbf{q}$ heat flux vector
q, q_1 heat flux in a one-dimensional flow of heat
$R_{\alpha m}$ coefficient defined by formula (55)
S entropy
T temperature
T_{a0} temperature amplitude at $x = 0$
$t = \tau c^2/\kappa$ dimensionless time for a case of one-dimensional flow of heat in semi-infinite region
U internal energy
x, y, z cartesian coordinates
$Y = x/l$ dimensionless coordinate in a case of one-dimensional flow of heat in finite region

Greek symbols
$\alpha = \kappa/cl$ dimensionless thermal diffusivity
κ thermal diffusivity
$\lambda = xc/\kappa$ dimensionless coordinate in the case of one-dimensional flow of heat in semi-infinite region
$\lambda_m = (2m-1)\pi$ eigenvalue
$\vartheta = \tau \mp x/c$ local time
ρ density
$\sigma = \vartheta c/l$ dimensionless local time
τ time
$\Omega = 2\omega\kappa/c^2$ dimensionless frequency
ω cyclic frequency
$\xi = q/c\rho C_v T_0$ dimensionless heat flux
$\Psi = T/T_0$ dimensionless temperature

Subscripts
A, B, E relates to thermodynamic bodies
D, D_1, D_2, DA, DB, D_1A, D_1E, D_2B, D_2E relates to thermal mediators
m integer
n vector component along $\mathbf{n}$
0 denotes reference values of characteristic parameters

2. Basic thermodynamic models of heat conduction

Two basic thermodynamic models of heat conduction may be introduced (Fig. 1). The first of them shown on Fig. 1a, assumes direct thermal contact of bodies. The second model presented in Fig. 1b, applies a thermal mediator concept.

2.1. Thermodynamic analysis of the thermal contact model

The thermodynamic method applied to the thermal contact model (Fig. 1a), revealed that, in this case, thermal propagation speed is infinite, and Fourier equation holds [15,16]. These conclusions are based on the following considerations. Let $T_A(\tau), U_A(\tau)$ and $S_A(\tau)$ be the body's A temperature, internal energy, and entropy, respectively, while $T_B(\tau)$, $U_B(\tau)$, $S_B(\tau)$ are respective values of the same parameters for the body B, which is in thermal contact with the body A (τ is time). It is assumed that every body has uniform temperature and constant volume. For an insulated thermodynamic system consisting of the bodies A and B, the First and Second Laws formulations are

$$\frac{dU_A}{d\tau} + \frac{dU_B}{d\tau} = 0 \tag{1}$$

$$\frac{dS_A}{d\tau} + \frac{dS_B}{d\tau} \geqslant 0 \tag{2}$$

After introducing heat flows $Q_A(\tau) = dU_A/d\tau$ and $Q_B(\tau) = dU_B/d\tau$ for the corresponding bodies, Eq. (1) may be rewritten as

$$Q_A(\tau) = -Q_B(\tau) \tag{3}$$

As $dS_A/d\tau = Q_A(\tau)/T_A(\tau)$ and $dS_B/dt = -Q_A(t)/T_B(\tau)$, inequality (2) is transformed into

$$Q_B(\tau)[T_A(\tau) - T_B(\tau)] \geqslant 0 \tag{4}$$

Expressions (3) and (4) define basic thermodynamic properties of any heat transfer process in the thermal contact model. Formula (3) proves that heat is instantaneously extracted from one body and supplied to another body. This means that in the thermal contact model does not exist any time lag between heat supply and heat extraction. Therefore the first basic property of any heat transfer process in the thermal contact model is infinite speed of heat propagation.

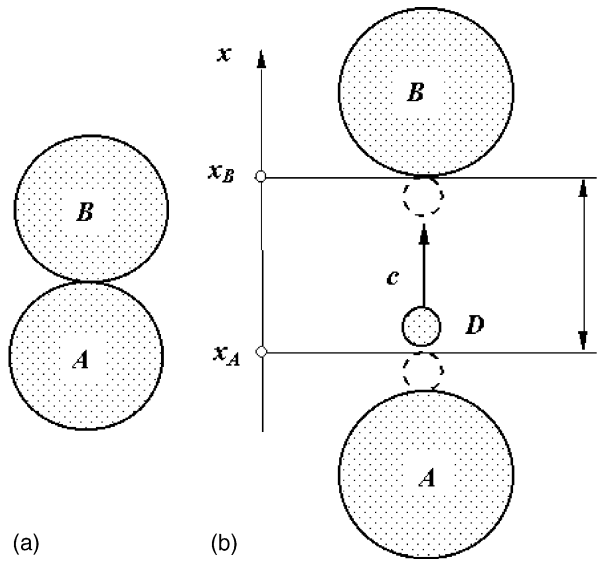

Fig. 1. Basic thermodynamic models of heat conduction: the thermal contact model (a), the model applying the thermal mediator concept (b).

Expression (4) represents the second basic property of heat conduction. Sufficient condition of this inequality validity is

$$Q_B(\tau) = H[T_A(\tau) - T_B(\tau)] \tag{5}$$

where H is a positive coefficient. Formula (5) shows that instantaneous heat flow in the analyzed process is affected by subsystems instantaneous temperatures difference corresponding to the same moment of time to which heat flow relates. Thus, it is proven that in the heat contact model no time lag exists between heat flow and temperature.

For an isotropic heat conductive continuum with constant heat conductivity and constant volume, Fourier law

$$\mathbf{q} = -K\nabla T \tag{6}$$

and Fourier equation

$$\frac{\partial T}{\partial \tau} = \kappa \nabla^2 T \tag{7}$$

are formulated by using only thermodynamic inequality (4), energy conservation equation

$$\rho C_v \frac{\partial T}{\partial \tau} + \nabla \cdot \mathbf{q} = 0 \tag{8}$$

and local thermodynamic equilibrium principle [15,16]. In expressions (6) and (7), $\mathbf{q}$ is heat flux vector; ρ and C_v are the continuum density and specific heat capacity, respectively; K and κ denote the continuum heat conductivity and heat diffusivity, respectively. Therefore it may be concluded that, for the thermal contact model, infinite speed of heat propagation, Fourier law and Fourier equation are direct corollaries of the First and the Second laws of thermodynamics, and local thermodynamic equilibrium principle.

2.2. Thermodynamic analysis of the model applying the thermal mediator concept

In the model applying the thermal mediator concept (Fig. 1b), A and B are constant volume thermodynamic bodies, which have uniform temperatures and are not in direct thermal contact, D is a thermal mediator. It is assumed that the thermal mediator is a physical object with the following properties:

1. The thermal mediator can be characterized by thermodynamic parameters: temperature T_D, internal energy U_D and entropy S_D.
2. It can move between bodies A and B with speed c.
3. The thermal mediator temperature satisfies conditions: $T_A > T_D > T_B$, or $T_A < T_D < T_B$ depending on the relation between temperatures T_A and T_B.
4. The thermal mediator participates in a three stage cyclic process: (a) it comes in thermal contact with one body; (b) then it moves to another body being adiabatically insulated; (c) it comes in thermal contact with another body reaching initial thermodynamic state which the thermal mediator had before the stage "a". As a result, a nondirect heat transfer between two bodies occurs with a time shift determined by the thermal mediator speed and the distance between the bodies.

According to formulated properties of thermal mediators, in the thermodynamic system consisting of subsystems A, B and D, the following process occurs:

1. At the moment of time τ thermal mediator D is in thermal contact with the body A, and infinitesimal heat transfer process lasting $d\tau$, takes place. For it, the First and Second Laws equations are

$$dU_A + dU_{DA} = 0 \quad (9)$$

$$\frac{dU_A}{T_A(\tau)} + \frac{dU_{DA}}{T_D} \geqslant 0 \quad (10)$$

where dU_{DA} is a differential increment of the thermal mediator internal energy.

2. The thermal mediator moves to the body B, this movement lasts $\Delta\tau = l/c$, where $l > 0$ is the distance. At this stage, the thermal mediator is adiabatically insulated: $dU_D = 0$, $dS_D = 0$.
3. At the moment of time $\tau + \Delta\tau = \tau + l/c$, the thermal mediator D comes in thermal contact with the body B, and infinitesimal heat transfer process lasting $d\tau$, occurs. At the end of this stage, the thermal mediator reaches its initial state which it had before the first stage of the process. The First and the Second laws equations for the third stage are

$$dU_B + dU_{DB} = 0 \quad (11)$$

$$\frac{dU_B}{T_B(\tau + l/c)} + \frac{dU_{DB}}{T_D} \geqslant 0 \quad (12)$$

where dU_{DB} is a differential increment of the thermal mediator internal energy, and the following thermodynamic cycling condition holds

$$dU_{DA} + dU_{DB} = 0 \quad (13)$$

After summarizing Eqs. (9) and (11) and taking into account formula (13), the final First Law expression is obtained

$$Q_A(\tau) = -Q_B(\tau + l/c) \quad (14)$$

where $Q_A = dU_a/d\tau$ and $Q_B = dU_B/d\tau$ are heat flows.

From expression (14) follows that in the analyzed model heat propagates with speed c.

By summarizing Eqs. (10) and (12) and using relation (13) it is found that

$$Q_B(\tau + l/c)[T_A(\tau) - T_B(\tau + l/c)] \geqslant 0 \quad (15)$$

$$Q_A(\tau)[T_B(\tau + l/c) - T_A(\tau)] \geqslant 0 \quad (16)$$

Sufficient conditions of these inequalities validity are

$$Q_B(\tau + l/c) = H_c[T_A(\tau) - T_B(\tau + l/c)] \quad (17)$$

$$Q_A(\tau) = H_c[T_B(\tau + l/c) - T_A(\tau)] \quad (18)$$

where H_c is a positive coefficient. It is obvious that expression (15) is equivalent to (16), and formula (17) is equivalent to (18).

These formulae show that in the analyzed model heat flow is affected by the temperature difference where the temperatures are taken with a time delay corresponding to the finite speed of heat propagation, and in this case the heat, as usually, always flows from the hot body to the cold body. Expressions (15)–(18) reflect fundamental temperatures difference-heat flow relations. They are corollaries of the First and Second laws of thermodynamics obtained in a purely deductive way.

Fig. 2 presents a more complex model of heat conduction than shown in Fig. 1, involving three constant volume bodies A, B, E and two thermal mediators D_1, D_2. The bodies A, B, E have respective temperatures T_A, T_B, T_E and internal energies U_A, U_B, U_E. Thermal mediators D_1, D_2 have respective temperatures T_{D_1}, T_{D_2} and

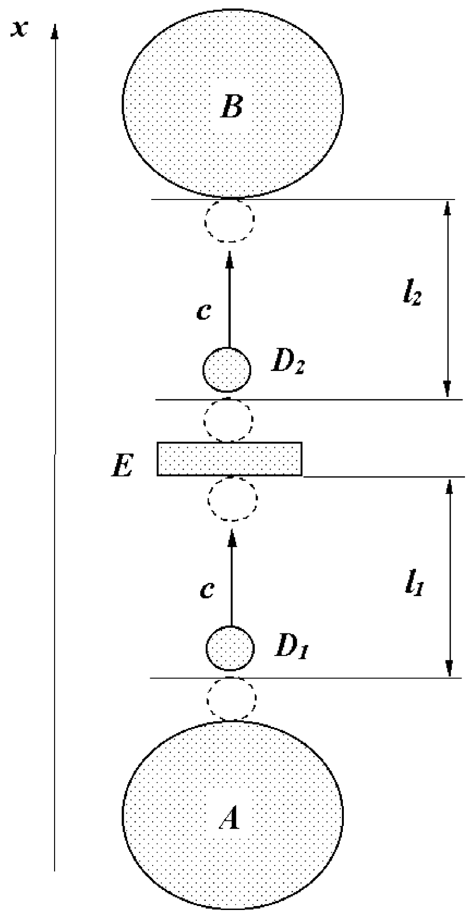

Fig. 2. Three body thermodynamic model of heat conduction applying the thermal mediator concept (A, B, E are bodies between whom heat transfer occurs; D_1, D_2 are the thermal mediators).

internal energies U_{D_1}, U_{D_2}. The thermal mediator's temperatures satisfy the following conditions: $T_A > T_{D_1} > T_E$, or $T_A < T_{D_1} < T_E$; $T_E > T_{D_2} > T_B$, or $T_E < T_{D_2} < T_B$ depending on direction of heat flows. In the three body model, a five stage process takes place:

1. At the moment of time τ thermal mediator D_1 is in thermal contact with the body A, and infinitesimal heat transfer process with duration $d\tau$ occurs

$$dU_A + dU_{D_1A} = 0 \tag{19}$$

where dU_{D_1A} is a differential increment of thermal mediator internal energy.

2. The thermal mediator D_1 moves to the body E during time interval $(\Delta\tau)_1 = l_1/c$, where $l_1 > 0$ is the distance. At this stage, thermal mediator is adiabatically insulated: $dU_{D_1} = 0$.
3. At the moment of time $\tau + \Delta\tau_1 = \tau + l_1/c$ thermal mediator D_1 comes in thermal contact with the body E. Instantaneously thermal mediator D_2 is also in thermal contact with E, and infinitesimal heat transfer process lasting $d\tau$ originates. At the end of this stage, the thermal mediator D_1 reaches its initial state which it had before the first stage of the process. For the third stage, the First law equation is

$$dU_E + dU_{D_1E} + dU_{D_2E} = 0 \tag{20}$$

where dU_{D_1E}, dU_{D_2E} are differential increments of the thermal mediators internal energy. The thermodynamic cycling condition requires

$$dU_{D_1A} + dU_{D_1E} = 0 \tag{21}$$

4. The thermal mediator D_2 moves to the body B during time interval $\Delta\tau_2 = l_2/c$, where $l_2 > 0$ is the distance. At this stage, thermal mediator is adiabatically insulated: $dU_{D_2} = 0$.
5. At the moment of time $\tau + \Delta\tau_1 + \Delta\tau_2 = \tau + (l/c)$, where $l = l_1 + l_2$, thermal mediator D_2 comes in thermal contact with the body B, and infinitesimal heat transfer process lasting $d\tau$ takes place. At the end of this stage, the thermal mediator D_2 reaches its initial state which it had before the third stage of the process. Therefore it is

$$dU_B + dU_{D_2B} = 0 \tag{22}$$

$$dU_{D_2E} + dU_{D_2B} = 0 \tag{23}$$

where dU_{D_2B} is differential increment of the thermal mediator internal energy.

As a result of summarizing Eqs. (19), (20), (22) and taking into account formulae (21), (23), the following energy balance equation holds for a case when a body E thermally interacts with two other bodies A and B

$$Q_A(\tau) + Q_B(\tau + l/c) + \left.\frac{dU_E}{d\tau}\right|_{\tau+l_1/c} = 0 \tag{24}$$

3. Kinetic and governing equations for heat conduction with finite speed of heat propagation

In this section results of the thermodynamic analyses which led to final Eqs. (15)–(18) and (24), are applied to an isotropic heat conductive continuum. It is assumed that the local thermodynamic equilibrium principle is applicable, and the continuum has constant volume.

3.1. Equations for one-dimensional flow of heat

As the first step, one-dimensional flow of heat is analyzed. As in the book of Carslaw and Jaeger [17], it is assumed that isothermal surfaces are planes perpendicular to x axis. It is also suggested that the direction of thermal mediators velocity coincides with direction of x axis. Let isothermal planes $T_B(x + l, \tau)$ and $T_A(x, \tau)$ represent bodies A and B, and an arbitrary cylindric volume between these planes is the body E. It is also suggested that the distance between these planes is infinitesimal $l = dx$. After simple calculations, formulae (16) and (24) may be replaced by

$$-q(x,\tau)\left(\frac{\partial T}{\partial x} + \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \geqslant 0 \tag{25}$$

$$\rho C_v \frac{\partial T}{\partial \tau} + \frac{\partial q}{\partial x} + \frac{1}{c}\frac{\partial q}{\partial \tau} = 0 \tag{26}$$

where $q(x,\tau)$, ρ, C_v are heat flux, density and specific heat capacity, respectively.

Inequality (25) is equivalent to the condition

$$q(x,\tau) = -K_c\left(\frac{\partial T}{\partial x} + \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \tag{27}$$

where K_c is a positive coefficient, which easily may be identified as the thermal conductivity of the continuum $K_c = K$.

If a coordinate system $x_1 = -x$ is applied, and thermal mediators direction of motion is left unchanged, Eqs. (26) and (27) are transformed into

$$\rho C_v \frac{\partial T}{\partial \tau} + \frac{\partial q_1}{\partial x_1} - \frac{1}{c}\frac{\partial q_1}{\partial \tau} = 0 \tag{28}$$

$$q_1(x_1,\tau) = -K\left(\frac{\partial T}{\partial x_1} - \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \tag{29}$$

This means that in the general case of one dimensional continuum, the propagation of thermal perturbation is governed by the following system of partial derivative equations, where the first of them is a kinetic equation—a corollary of the Fist and Second Laws, and the second is a corollary of the First Law

$$q(x,\tau) = -K\left(\frac{\partial T}{\partial x} \pm \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \tag{30}$$

$$\rho C_v \frac{\partial T}{\partial \tau} + \frac{\partial q}{\partial x} \pm \frac{1}{c}\frac{\partial q}{\partial \tau} = 0 \tag{31}$$

These formulae prove that heat propagates with finite speed c, while sign "+" relates to a case when heat propagates in direction of x axis, and sign "−" is applied to a case when heat propagates in opposite direction.

The derivatives $\partial q/\partial x$ and $\partial q/\partial \tau$ may be determined from Eq. (30) and substituted in Eq. (31), thus yielding the following governing equation for one-dimensional flow of heat in a case of constant thermal conductivity of the continuum

$$\frac{1}{\kappa}\frac{\partial T}{\partial \tau} - \frac{1}{c}\left(\frac{1}{c}\frac{\partial^2 T}{\partial \tau^2} \pm 2\frac{\partial^2 T}{\partial \tau \partial x}\right) - \frac{\partial^2 T}{\partial x^2} = 0 \tag{32}$$

where $\kappa = K/\rho C_v$ is thermal diffusivity.

3.2. Irreversibility of heat conduction and direction of thermal perturbations propagation

Eqs. (30)–(32) are purely phenomenological and describe the heat conduction from macroscopic, thermodynamic point of view. They do not depend on physical nature of thermal mediators. The speed c is the only thermal mediators' property which is present in the equations and is essential for heat conduction process. Therefore these equations are applicable to any heat conducting physical continuum—gaseous, liquid or solid. In every case physical nature of heat mediators is different. For instance, in crystal structures, phonon gas particles serve as heat mediators, while in a gaseous substances molecules of gas propagate thermal perturbations. In all cases, the velocity of heat propagation c is close to the speed of sound in the continuum, as it was assumed by Morse and Feshbach for gases [5]. More detailed picture of the thermal mediator concept may be developed by statistical methods.

Eqs. (30)–(32) are different depending on direction in which heat propagates. Therefore at any point instantaneously may exist only one thermal wave moving in certain direction. The governing equations reject interference and reflection of thermal waves. This feature is a consequence of irreversibility of heat conduction which absolutely excludes any possibility of reversing the process. Particularly it means that a thermal wave cannot reflect from the boundary and interfere with the wave moving to the boundary, thus generating temperatures higher than the highest temperature of interacting bodies or lower than the lowest temperature of interacting bodies. Because of this, the whole domain where heat conduction occurs, is divided on separate sub-domains. In every such sub-domain the direction of heat propagation does not change. These sub-domains are determined by initial and boundary conditions. The governing principles of localizing the sub-domains are:

Case 1. *A heat conductive semi-infinite domain in thermal equilibrium. Thermal perturbation on its boundary*. In this case, thermal perturbation wave propagates from the boundary into the domain.

Case 2. *A bounded heat conductive space domain in thermal equilibrium. Thermal perturbations on its boundaries*. The thermal perturbation waves propagate from the boundaries into the domain until they meet inside the domain. The surfaces where the waves meet, divide the domain on sub-domains.

Case 3. *A nonuniform temperature field in the space domain*. In this case, the space sub-domains, where direction of heat propagation does not change, already exist initially. Their borders are formed at a starting moment, when thermal equilibrium exists in the domain, and thermal perturbations are introduced at the borders of the domain or inside domain.

3.3. Equations for three dimensional isotropic continuum

In a three dimensional case of an isotropic continuum, the only physically preferable direction is the direction of the temperature gradient vector ∇T, and the condition $\mathbf{q} \times \mathbf{n} = 0$ holds, where $\mathbf{n} = \nabla T/|\nabla T|$ denotes the temperature gradient unit vector, $\mathbf{q}$ is the heat flux vector, and sign $|\ |$ denotes vector magnitude. For the $\mathbf{n}$-direction, equations similar to (30) and (31) may be locally applied

$$q_n(x,y,z,\tau) = -K\left(\frac{\partial T}{\partial n} \pm \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \tag{33}$$

$$\rho C_v \frac{\partial T}{\partial \tau} + \frac{\partial q_n}{\partial n} \pm \frac{1}{c}\frac{\partial q_n}{\partial \tau} = 0 \tag{34}$$

where sign "+" relates to a case when the heat perturbation propagates in $\mathbf{n}$ direction, and sign "−" characterizes the case when the heat perturbation propagates in the opposite direction; x, y, z are cartesian coordinates of a point in the continuum; $q_n(x,y,z,\tau)$ is a component of heat flux vector $\mathbf{q}(x,y,z,\tau)$ along $\mathbf{n}$.

As $\partial T/\partial n = |\nabla T|$, $\mathbf{q} = q_n\mathbf{n}$, and $\partial q_n/\partial n = \nabla \cdot \mathbf{q}$, from expressions (33) and (34) follows that

$$q_n(x,y,z,\tau) = -K\left(|\nabla T| \pm \frac{1}{c}\frac{\partial T}{\partial \tau}\right) \tag{35}$$

$$\mathbf{q} = -K\nabla T\left(1 \pm \frac{1}{c|\nabla T|}\frac{\partial T}{\partial \tau}\right) \tag{36}$$

$$\rho C_v \frac{\partial T}{\partial \tau} + \nabla \cdot \mathbf{q} \pm \frac{1}{c}\frac{\partial q_n}{\partial \tau} = 0 \tag{37}$$

Nonlinear partial differential equations (35)–(37) describe heat conduction in three dimensional isotropic continuum with finite speed of heat perturbations propagation. Their nonlinearity is a very essential obstacle for obtaining analytical solutions.

4. Basic properties of derived equations for one-dimensional flow of heat

4.1. The governing equation type and finite speed of heat propagation

Eq. (32) which governs one-dimensional flow of heat with finite speed of thermal perturbations propagation, is a particular case of the general two-dimensional partial differential equation [5]

$$A(x,\tau)\frac{\partial^2 U}{\partial x^2}+2B(x,\tau)\frac{\partial^2 U}{\partial x\partial\tau}+D(x,\tau)\frac{\partial^2 U}{\partial\tau^2} = E\left(x,\tau,U,\frac{\partial U}{\partial x},\frac{\partial U}{\partial\tau}\right) \quad (38)$$

For Eq. (32) $A(x,\tau)=-1$, $B(x,\tau)=\mp 1/c$, $D(x,\tau)=-1/c^2$, and $B^2(x,\tau)-A(x,\tau)D(x,\tau)=0$. It means that equation (32) is of parabolic type. Because of this, formulae (30)–(32) may be simplified by introducing a new independent variable $\vartheta=\tau\mp x/c$ instead of τ. In formula for ϑ, the sign "−" relates to a case when thermal perturbation propagates in x direction, while the sign "+" means that thermal perturbation propagates in the opposite direction. After such a substitution of ϑ in expressions (30)–(32), the following equations determining $q(x,\vartheta)$ and $T(x,\vartheta)$, are obtained:

$$q(x,\vartheta)=-K\frac{\partial T}{\partial x} \quad (39)$$

$$\rho C_v\frac{\partial T}{\partial\vartheta}+\frac{\partial q}{\partial x}=0 \quad (40)$$

$$\frac{\partial T}{\partial\vartheta}=\kappa\frac{\partial^2 T}{\partial x^2} \quad (41)$$

Obviously Eq. (39) is similar to the regular Fourier law equation (6); Eq. (40) is similar to regular energy balance equation (8); and (41) is similar to Fourier governing equation (7). But there is also a principal difference. In regular heat conduction equations time τ and space coordinate x are independent variables, while in the analyzed case, $\vartheta=\tau\mp x/c$ and x are independent variables. Because of this, governing equation (41) predicts quite different from Fourier equation solutions describing irreversible heat conduction with finite speed of heat propagation.

Solutions of the regular Fourier equation (7) predict that any thermal perturbation introduced at a time moment $\tau=\tau_0$ in a point $x=x_0$ instantaneously, at the same moment of time $\tau=\tau_0$ affects all space domain.

In the derived governing parabolic equation (41), a new independent variable $\vartheta=\tau\mp x/c$, instead of time τ, determines its solutions. Therefore any thermal perturbation introduced at the time moment $\tau=\tau_0$ and in the point $x=x_0$ corresponds to a value of the independent variable $\vartheta_0=\tau_0\mp x_0/c$. At the same value of the independent variable $\vartheta_0=\tau\mp x/c$, the thermal perturbation affects all space domain, because governing equation (41) is of parabolic type. Therefore the following formula holds

$$\vartheta_0=\tau\mp x/c=\tau_0\mp x_0/c \quad (42)$$

It means that

$$\tau-\tau_0=|x-x_0|/c \quad (43)$$

where symbol | | denotes absolute value.

Formula (43) shows that the thermal perturbation with a time delay reaches different points of the continuum. This time delay is determined by the distance $|x-x_0|$ and value of c. Thus it is proven that c is speed of thermal perturbation propagation, and derived parabolic governing equation describes heat conduction with finite speed of heat propagation.

In order to reach a clearer understanding of properties of Eq. (41), the following conditions are assumed:

1. The continuum is semi-infinite; it is bounded by a plane at $x=0$, and the points of the continuum have positive x.
2. Initial temperature, corresponding to $\vartheta\leqslant 0$, is zero.
3. At $x=0$ a thermal perturbation $T(0,\vartheta)$ $(\vartheta\geqslant 0)$ is introduced. It means, that thermal perturbations propagate in x direction, therefore it is $\vartheta=\tau-x/c$.

Let the function $T_0(x,\vartheta)$ be respective solution of Eq. (41) valid for $\vartheta>0$. For an arbitrary moment of time $\tau_0\geqslant 0$, exists such an isothermal plane with a coordinate $x_0=c\tau_0$ where $\vartheta=\tau_0-x_0/c=0$. At the same moment of time $\tau=\tau_0$, for $x<x_0$, there is $\vartheta=-((x-x_0)/c)>0$, and the temperature is determined by the thermal perturbation, i.e. by the function $T_0(x,\vartheta)$. For the same time moment $\tau=\tau_0$ and $x\geqslant x_0$, there is $\vartheta=-((x-x_0)/c)\leqslant 0$, and the points of the continuum have initial zero temperature. The boundary between the region of continuum affected by thermal perturbation, and the region with zero initial temperature moves with the speed c. At the beginning of the process, there is $\tau_0=0$, $x_0=0$, and continuum is not affected by thermal perturbation. Thus the derived parabolic governing equation (41) describes heat conduction with finite speed of heat propagation c. In the analyzed case, physical meaning of $\vartheta=\tau-x/c$ becomes clear: it is the local time at a given point counted from a moment, when thermal perturbation arrives at this point. When $x/c\tau\ll 1$, $\vartheta\approx\tau$, and Fourier equation holds.

Fig. 3 further illustrates one-dimensional flow of heat in a semi-infinite region with $x \geqslant 0$, $T(x,0) = 0$, $T(0,\vartheta) = T_0 = \text{const}$. In this case, the heat perturbation propagates in x direction. Dimensionless quantities: time $t = \tau c^2/\kappa$, coordinate $\lambda = xc/\kappa$, temperature $\Psi = T/T_0$, and heat flux $\xi = q/c\rho C_v T_0$, are introduced. It follows from Eqs. (39) and (41), that for $\lambda < t$ dimensionless temperature and heat flux are determined as

$$\Psi = erfc\left(\frac{\lambda}{2\sqrt{t-\lambda}}\right) \tag{44}$$

$$\xi = \frac{1}{\sqrt{\pi(t-\lambda)}}\exp\left[-\frac{\lambda^2}{4(t-\lambda)}\right] \tag{45}$$

while for $\lambda \geqslant t$ there are $\Psi = 0$, $\xi = 0$. For comparison, the same variables determined from Fourier equation are shown in Fig. 3, thus confirming that the derived equations define wave propagation of thermal perturbations.

4.2. Irreversibility of heat conduction with finite speed of heat propagation

Parabolic character of the governing equation (41) reflects absolute irreversibility of heat conduction—a feature which is common with Fourier equation (7). This leads to the following peculiarities of its solutions which are similar to Fourier equation [5]:

1. The solutions derived for positive and negative times are principally different.
2. All irregularities in the solution of the Eq. (41) are smoothing with time. Therefore during heat propagation, a sharp temperature front in heat conductive continuum is physically impossible. Only artificially at initial conditions it can be introduced.
3. In a bounded closed continuum, the highest and the lowest values of temperature are reached either on boundaries of the continuum or at initial conditions. This theorem is valid only for a parabolic equation of the type (41) without the heat source term, and does not hold for hyperbolic type governing equation. This means that Eq. (41) is consistent with the Second Law of thermodynamics and reflects irreversibility of heat transfer process according to which heat always flows from hot to cold bodies. The principle of maximal and minimal values is not valid for hyperbolic governing equation. Because of this, hyperbolic heat conduction equation has nonphysical solutions where heat flows from cold to hot bodies.

4.3. Periodical solutions of the derived governing equation

Additional information about properties of governing equation (41) brings analysis of a case when a periodical temperature perturbation is introduced at the border of semi-infinite domain. It is assumed that $x \geqslant 0$, $T(x,0) = 0$, $T(0,\vartheta) = T_{a0}\sin(\omega\vartheta)$. Appropriate periodical solution of Eq. (41) expressed in variables x and τ, is

$$T(x,\tau) = T_{a0}\exp\left[-\left(\frac{\omega}{2\kappa}\right)^{1/2}x\right] \times \sin\left[\omega\tau - \left(\frac{\omega}{c} + \left(\frac{\omega}{2\kappa}\right)^{1/2}\right)x\right] \tag{46}$$

This is an equation of a sinusoidal temperature wave traveling in x direction. In this wave, amplitude of temperature oscillations monotonically decreases. The

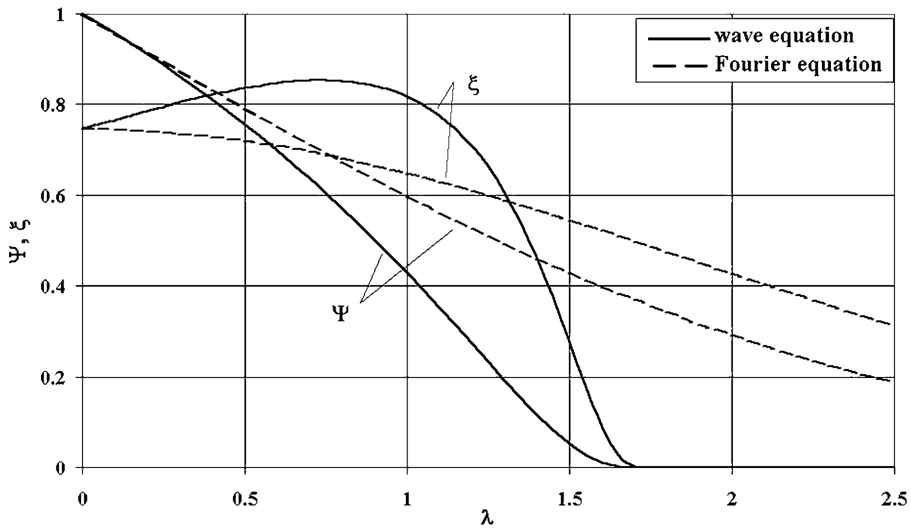

Fig. 3. Temperature and heat flux profiles for $t = 1.8$ defined by the derived and Fourier equations.

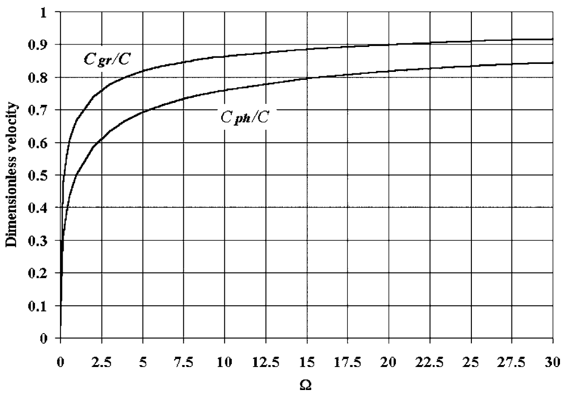

Fig. 4. The wave dimensionless phase c_{ph}/c and group c_{gr} velocities as functions of $\Omega = 2\omega\kappa/c^2$, determined from derived heat conduction governing equation.

wave number k, phase velocity c_{ph}, and group velocity c_{gr} are defined as

$$k = \frac{\omega}{c} + \left(\frac{\omega}{2\kappa}\right)^{1/2} \tag{47}$$

$$c_{ph} = \frac{\omega}{k} = \left(\frac{1}{c} + \frac{1}{\sqrt{2\omega\kappa}}\right)^{-1} \tag{48}$$

$$c_{gr} = \frac{d\omega}{dk} = \left(\frac{1}{c} + \frac{1}{2\sqrt{2\omega\kappa}}\right)^{-1} \tag{49}$$

Formulae (48) and (49) prove that the wave is of dispersion type, and its phase and group velocities are finite, being always smaller than c: $c_{ph} < c_{gr} < c$. Fig. 4 illustrates how dimensionless phase and group velocities depend on dimensionless frequency $\Omega = 2\omega\kappa/c^2$.

Therefore it may be concluded that derived equations describe irreversible heat conduction phenomenon with finite speed of heat propagation.

5. Comparison of solutions of the derived governing equation, Fourier equation and Hyperbolic equation

Comparison of temperature distributions predicted by the derived governing equation (41), Fourier equation (7) and hyperbolic Cattaneo equation

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial \tau^2} + \frac{1}{\kappa}\frac{\partial T}{\partial \tau} - \frac{\partial^2 T}{\partial x^2} = 0 \tag{50}$$

brings additional information regarding heat conduction with finite speed of thermal perturbations propagation. The temperature distributions are calculated for one-dimensional flow of heat in one dimensional continuum bounded by two parallel planes at $x = 0$ and $x = l$, the initial condition $T(x, 0) = 0$ and symmetrical boundary conditions $T(0, \tau) = T(l, \tau) = 1$. For the Cattaneo equation, additional initial condition $\frac{\partial T}{\partial \tau}\big|_{\tau=0} = 0$ is applied. Dimensionless variables $a = c\tau/l$ and $Y = x/l$ are introduced. The solutions are obtained by separation of variables.

In the case of derived parabolic governing equation (41), two thermal perturbations waves proceed. One is moving from the boundary $Y = 0$ in Y direction, another is propagating from the boundary $Y = 1$ in opposite direction. Accordingly two sub-domains exist. One sub-domain occupies a region with $0 \leqslant Y < 0.5$, another sub-domain is located at $0.5 < Y \leqslant 1$. For the first sub-domain, solution of the governing equation (41) is

$$T = 1 + \sum_{m=1}^{\infty} A_m \sin(\lambda_m Y) \exp(-\lambda_m^2 \alpha\sigma) \tag{51}$$

where $\lambda_m = (2m-1)\pi$ is eigenvalue; $A_m = -4/(2m-1)\pi$ denotes Fourier coefficient; $a = c\tau/l$ and $\sigma = a - Y$ are dimensionless time and local time, respectively; $\alpha = \kappa/cl$ denotes dimensionless thermal diffusivity. If $a \leqslant 0.5$, formula (51) is applicable only for $Y \leqslant a$, while for $Y > a, T = 0$. If $a > 0.5$, expression (51) determines temperature field for all values of $Y < 0.5$.

For the second sub-domain, Y must be replaced by $1 - Y$ in formula (51). In this case, if $a \leqslant 0.5$, formula (51) is valid for $1 - Y \leqslant a$, while for $1 - Y > a$, $T = 0$. When $a > 0.5$, expression (51) is applicable for all values of $0.5 < Y \leqslant 1$.

For Fourier equation, the solution of the problem is [17]:

$$T = 1 + \sum_{n=1}^{\infty} A_m \sin(\lambda_m Y) \exp(-\lambda_m^2 \alpha a) \tag{52}$$

For Cattaneo equation, various forms of the solution are published [6,9]. Another form of the solution may be presented as

$$T = 1 + \exp\left(-\frac{a}{2\alpha}\right) \sum_{n=1}^{\infty} A_m D_m \sin(\lambda_m Y) \tag{53}$$

where the coefficient D_m is defined as:

(a) For $1 - 4\lambda_m^2\alpha^2 > 0$

$$D_m = \frac{1}{1 - R_{\alpha m}}\left[\exp\left(\frac{\sqrt{1-4\lambda_m^2\alpha^2}}{2\alpha}a\right) - R_{\alpha m}\exp\left(-\frac{\sqrt{1-4\lambda_m^2\alpha^2}}{2\alpha}a\right)\right] \tag{54}$$

$$R_{\alpha m} = \frac{1 - \sqrt{1-4\lambda_m^2\alpha^2}}{1 + \sqrt{1-4\lambda_m^2\alpha^2}} \tag{55}$$

(b) For $1 - 4\lambda_m^2\alpha^2 < 0$

$$D_m = \cos(\Lambda_m a) + \frac{1}{2\alpha\Lambda_m}\sin(\Lambda_m a) \tag{56}$$

$$\Lambda_m = \frac{1}{2\alpha}\sqrt{4\lambda_m^2\alpha^2 - 1} \tag{57}$$

Fig. 5 presents transient temperature distributions obtained by using derived formulae. They are symmetrical with respect to the line $Y = 0.5$. Though the "Parabolic wave" and "Hyperbolic" curves display wave propagation of thermal perturbations, their behavior is quite different. The first one is smooth, while the second curve has a sharp front.

Further development of the temperature fields is shown in Fig. 6. As in the Fig. 5, temperature distribution curves are symmetrical with respect to the line $Y = 0.5$. "Parabolic wave" curve continues to be smooth at all values of $Y \neq 0$, and has a nonzero derivative at $Y = 0.5$, because there is a zero heat flux

$$q(x,\tau) = -K\left(\frac{\partial T}{\partial x} \pm \frac{1}{c}\frac{\partial T}{\partial \tau}\right) = 0 \tag{58}$$

At this point, $\partial T/\partial\tau > 0$, and at $Y < 0.5$ formula (58) includes sign "+", while at $Y > 0.5$ it has sign "−". Therefore at $Y = 0.5$, the temperature distribution for $Y < 0.5$ has a nonzero negative space temperature derivative $\partial T/\partial Y < 0$, and the distribution for $Y > 0.5$ has a nonzero positive derivative $\partial T/\partial Y > 0$.

In Fig. 6, "Hyperbolic" curve has a sharp front at $Y = 0$, and everywhere the temperatures are higher than the boundary temperature and initial temperature. These data confirm the findings of previous works [7,9,10], and show that, according to Cattaneo equation, heat may be transferred directly from the cold to the hot body. Such a feature of hyperbolic governing equation makes this equation nonconsistent with classical thermodynamics.

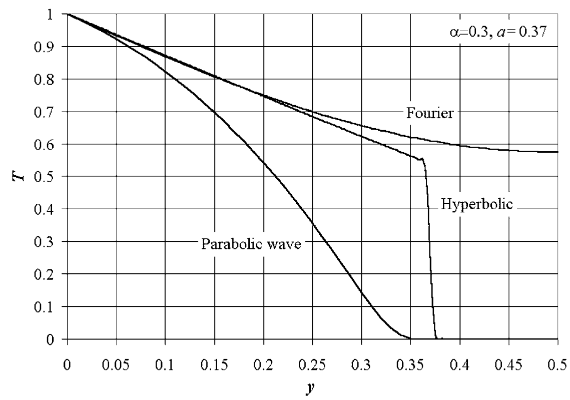

Fig. 5. Transient temperature distributions determined from the derived governing equation ("Parabolic wave"), Fourier equation and hyperbolic Cattaneo equation for $\alpha = 0.3$, $a = 0.37$.

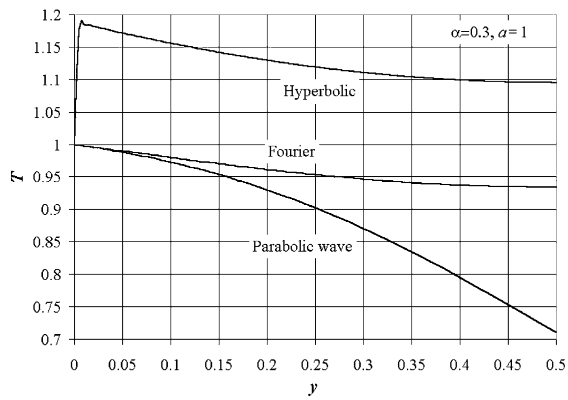

Fig. 6. Transient temperature distributions determined from the derived governing equation ("Parabolic wave"), Fourier equation and hyperbolic Cattaneo equation for $\alpha = 0.3$, $a = 1$.

6. Conclusions

A method of deriving thermodynamically consistent equations for heat conduction with finite speed of heat propagation is developed. It is based on direct application of the First and Second laws equations to an appropriate thermodynamic model.

Two thermodynamic models of heat conduction are introduced and studied. One of them—thermal contact model—is characterized by infinite speed of heat propagation. Another model based on the thermal mediator concept describes heat conduction with finite speed of heat propagation.

For a one-dimensional flow of heat with finite speed of heat propagation, the governing equation is linear and of parabolic type. In a three dimensional case of an isotropic continuum, conduction of heat with finite speed of heat propagation is described by a nonlinear system of equations.

References

[1] M.N. Özişik, D.Y. Tzou, On the wave theory in heat conduction, Trans. ASME. J. Heat Transfer 116 (1994) 526–535.
[2] M.B. Rubin, Hyperbolic heat conduction and the second law, Int. J. Eng. Sci. 30 (11) (1992) 1665–1676.
[3] C. Baik, A.S. Lavine, On hyperbolic heat conduction equation and the second law of thermodynamics, Trans. ASME. J. Heat Transfer 117 (1995) 256–263.
[4] M. Grmela, G. Lebon, Finite-speed propagation of heat: a nonlocal and nonlinear approach, Physica A 248 (3) (1998) 428–441.
[5] P.M. Morse, H. Feshbach, Methods of Theoretical Physics, McGraw-Hill Publishing Company, New York, 1953, pp.137–138, 173–174, 676–692, 865–869, 1584–1592.
[6] A. Barletta, E. Zanchini, Hyperbolic heat conduction and local equilibrium: A second law analysis, Int. J. Heat Mass Transfer 40 (5) (1997) 1007–1016.

[7] C. Körner, H.W. Bergmann, The physical defects of the hyperbolic heat conduction equation, Appl. Phys. A. Mater. Sci. Process. 67 (4) (1998) 397–401.

[8] E. Zanchini, Hyperbolic heat conduction theories and nondecreasing entropy, Phys. Rev. B. Condens. Matter Mater. Phys. 60 (2) (1999) 991–997.

[9] Y. Taitel, On the parabolic, hyperbolic and discrete formulation of the heat conduction equation, Int. J. Heat Mass Transfer 15 (1972) 369–371.

[10] A. Haji-Sheikh, W.J. Minkovycz, E.M. Sparrow, Certain anomalies in the analysis of hyperbolic heat conduction, Trans. ASME. J. Heat Transfer 124 (2002) 307–319.

[11] I. Muller, T. Ruggeri, Rational Extended Thermodynamics, second ed., Springer, New York, 1998.

[12] I. Muller, Speeds of Propagation in Classical and Relativistic Extended Thermodynamics, Living Reviews in Relativity, Published by Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Germany, 2001, (www.livingreviews.org/Articles/Volume2/1999-Imueller).

[13] D. Jou, J. Casas-Vazquez, G. Lebon, Extended Irreversible Thermodynamics, second ed., Springer, Berlin, Heidelberg, New York, 1996.

[14] M. Planck, Theory of Heat, The MacMillan Company, New York, 1949.

[15] I. Shnaid, Thermodynamical approach for the determination of the speed of heat propagation in heat conduction, in: The 27th Israel Conference on Mechanical Engineering. Conference Proceedings, Haifa, Israel, 1998, pp. 401–403.

[16] I. Shnaid, Thermodynamical proof of transport phenomena kinetic equations, J. Mech. Behavior Mater. 11 (5) (2000) 353–364.

[17] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, London, 1959, pp. 50, 96–97.